Exercise 4H

1 a $y = \frac{3}{4}x \Rightarrow \frac{dy}{dx} = \frac{3}{4}$

Using $S = 2\pi \int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$ gives:

$$S = 2\pi \int_4^8 \frac{3}{4}x\sqrt{1+\left(\frac{3}{4}\right)^2}\, dx$$

$$= \frac{3}{2}\pi \int_4^8 x\sqrt{1+\frac{9}{16}}\, dx$$

$$= \frac{3}{2}\pi \int_4^8 x\sqrt{\frac{25}{16}}\, dx$$

$$= \frac{3}{2}\pi \int_4^8 \frac{5}{4}x\, dx$$

$$= \frac{15}{8}\pi \int_4^8 x\, dx$$

$$= \frac{15}{8}\pi \left[\frac{1}{2}x^2\right]_4^8$$

$$= \frac{15}{16}\pi(64-16)$$

$$= 45\pi$$

1 b $x=\frac{4}{3}y \Rightarrow \frac{dx}{dy}=\frac{4}{3}$

When $x=4$, $y=3$

When $x=8$, $y=6$

Using $S=2\pi\int_{x_A}^{x_B} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ gives:

$$S=2\pi\int_3^6 \frac{4}{3}y\sqrt{1+\left(\frac{4}{3}\right)^2}\,dy$$

$$=\frac{8}{3}\pi\int_3^6 y\sqrt{1+\frac{16}{9}}\,dy$$

$$=\frac{8}{3}\pi\int_3^6 y\sqrt{\frac{25}{9}}\,dy$$

$$=\frac{8}{3}\pi\int_3^6 \frac{5}{3}y\,dy$$

$$=\frac{40}{9}\pi\int_3^6 y\,dy$$

$$=\frac{40}{9}\pi\left[\frac{1}{2}y^2\right]_3^6$$

$$=\frac{20}{9}\pi(36-9)$$

$$=60\pi \text{ as required}$$

Note that this can also be solved correctly using $S=2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

2 $y=x^3 \Rightarrow \frac{dy}{dx}=3x^2$

Using $S=2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ gives:

$$S=2\pi\int_0^1 x^3\sqrt{1+\left(3x^2\right)^2}\,dx$$

$$=2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx$$

$$=\frac{2\pi}{\frac{3}{2}\times 36}\left[\left(1+9x^4\right)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{\pi}{27}\left[(1+9)^{\frac{3}{2}}-(1+0)^{\frac{3}{2}}\right]$$

$$=\frac{\pi}{27}\left(10\sqrt{10}-1\right)$$

3 $y=\frac{1}{2}x^2$

$y=\frac{1}{2}x^2 \Rightarrow x=\pm\sqrt{2}y^{\frac{1}{2}} \Rightarrow \frac{dx}{dy}=\pm\frac{\sqrt{2}}{2}y^{-\frac{1}{2}}$

When $x=0$, $y=0$
When $x=2$, $y=2$

Using $S=2\pi\int_{x_A}^{x_B} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ gives:

$$S=2\pi\int_0^2 \sqrt{2}y^{\frac{1}{2}}\sqrt{1+\left(\frac{\sqrt{2}}{2}y^{-\frac{1}{2}}\right)^2}\,dy$$

$$=2\sqrt{2}\pi\int_0^2 y^{\frac{1}{2}}\sqrt{1+\frac{1}{2y}}\,dy$$

$$=2\sqrt{2}\pi\int_0^2 y^{\frac{1}{2}}\sqrt{\frac{2y+1}{2y}}\,dy$$

$$=2\pi\int_0^2 y^{\frac{1}{2}}\sqrt{\frac{2y+1}{y}}\,dy$$

$$=2\pi\int_0^2 \sqrt{2y+1}\,dy$$

$$=\frac{2\pi}{\frac{3}{2}\times 2}\left[(2y+1)^{\frac{3}{2}}\right]_0^2$$

$$=\frac{2\pi}{3}\left((4+1)^{\frac{3}{2}}-(0+1)^{\frac{3}{2}}\right)$$

$$=\frac{2\pi}{3}\left(5\sqrt{5}-1\right)$$

Note that this can also be solved correctly using $S=2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

4 $y^2 = 16x \Rightarrow y = \pm 4x^{\frac{1}{2}} \Rightarrow \dfrac{dy}{dx} = \pm 2x^{-\frac{1}{2}}$

Using $S = 2\pi\displaystyle\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ gives:

$$S = 2\pi\int_5^{12} 4x^{\frac{1}{2}}\sqrt{1+\left(2x^{-\frac{1}{2}}\right)^2}\,dx$$

$$= 8\pi\int_5^{12} x^{\frac{1}{2}}\sqrt{1+\frac{4}{x}}\,dx$$

$$= 8\pi\int_5^{12} x^{\frac{1}{2}}\sqrt{\frac{x+4}{x}}\,dx$$

$$= 8\pi\int_5^{12} \sqrt{x+4}\,dx$$

$$= 8\pi\times\frac{2}{3}\left[(x+4)^{\frac{3}{2}}\right]_5^{12}$$

$$= \frac{16\pi}{3}\left[(12+4)^{\frac{3}{2}}-(5+4)^{\frac{3}{2}}\right]$$

$$= \frac{16\pi}{3}(64-27)$$

$$= \frac{592}{3}\pi$$

5 a $y = \cosh x \Rightarrow \dfrac{dy}{dx} = \sinh x$

Using $S = 2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx$ gives:

$$S = 2\pi\int_0^1 \cosh x\sqrt{1+(\sinh x)^2}\,dx$$
$$= 2\pi\int_0^1 \cosh x\sqrt{1+\sinh^2 x}\,dx$$
$$= 2\pi\int_0^1 \cosh x\sqrt{\cosh^2 x}\,dx$$
$$= 2\pi\int_0^1 \cosh^2 x\,dx$$
$$= 2\pi\int_0^1 \frac{\cosh 2x+1}{2}\,dx$$
$$= \pi\int_0^1 (\cosh 2x+1)\,dx$$
$$= \pi\left[\frac{1}{2}\sinh 2x + x\right]_0^1$$
$$= \pi\left[\frac{1}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)+x\right]_0^1$$
$$= \pi\left[\frac{1}{4}\left(e^{2x}-e^{-2x}\right)+x\right]_0^1$$
$$= \pi\left[\left(\frac{1}{4}\left(e^{2}-e^{-2}\right)+1\right)-\left(\frac{1}{4}\left(e^{0}-e^{0}\right)+0\right)\right]$$
$$= \pi\left[\left(\frac{1}{4}\left(e^{2}-\frac{1}{e^2}\right)+1\right)\right]$$
$$= 8.84 \text{ (3 s.f.)}$$

5 b $y = \cosh x \Rightarrow \dfrac{dy}{dx} = \sinh x$

Using $S = 2\pi \int_{x_A}^{x_B} x\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx$ gives:

$$S = 2\pi \int_0^1 x\sqrt{1+(\sinh x)^2}\, dx$$
$$= 2\pi \int_0^1 x\sqrt{1+\sinh^2 x}\, dx$$
$$= 2\pi \int_0^1 x\sqrt{\cosh^2 x}\, dx$$
$$= 2\pi \int_0^1 x\cosh x\, dx$$

Let $u = x \Rightarrow du = dx$

$\dfrac{dv}{dx} = \cosh x \Rightarrow v = \sinh x$

$$S = 2\pi \left([x\sinh x]_0^1 - \int_0^1 \sinh x\, dx \right)$$
$$= 2\pi \left([x\sinh x]_0^1 - [\cosh x]_0^1 \right)$$
$$= 2\pi \left(\sinh(1) - \cosh(1) + \cosh(0) \right)$$
$$= 2\pi \left(\frac{e^1 - e^{-1}}{2} - \frac{e^1 + e^{-1}}{2} + \frac{e^0 + e^0}{2} \right)$$
$$= 2\pi \left(-e^{-1} + 1 \right)$$
$$= 2\pi \left(1 - \frac{1}{e} \right)$$
$$= 2\pi \left(\frac{e-1}{e} \right) \text{ as required}$$

6 a $y=\frac{1}{2x}+\frac{x^3}{6} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2x^2}+\frac{x^2}{2}$

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}=\sqrt{1+\left(-\frac{1}{2x^2}+\frac{x^2}{2}\right)^2}$$
$$=\sqrt{1+\left(\frac{1}{4x^4}+\frac{x^4}{4}-\frac{1}{2}\right)}$$
$$=\sqrt{\left(\frac{1}{4x^4}+\frac{x^4}{4}+\frac{1}{2}\right)}$$
$$=\sqrt{\frac{1}{4}\left(\frac{1}{x^4}+x^4+2\right)}$$
$$=\frac{1}{2}\sqrt{\left(\frac{1}{x^2}+x^2\right)^2}$$
$$=\frac{1}{2}\left(\frac{1}{x^2}+x^2\right) \text{ as required}$$

b Using $S=2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ gives:

$$S=\pi\int_1^3\left(\frac{1}{2x}+\frac{x^3}{6}\right)\left(x^2+\frac{1}{x^2}\right)\mathrm{d}x$$
$$=\pi\int_1^3\left(\frac{x}{2}+\frac{1}{2x^3}+\frac{x^5}{6}+\frac{x}{6}\right)\mathrm{d}x$$
$$=\frac{\pi}{2}\int_1^3\left(x+\frac{1}{x^3}+\frac{x^5}{3}+\frac{x}{3}\right)\mathrm{d}x$$
$$=\frac{\pi}{2}\left[\frac{1}{2}x^2-\frac{1}{2x^2}+\frac{x^6}{18}+\frac{x^2}{6}\right]_1^3$$
$$=\frac{\pi}{2}\left[\left(\frac{9}{2}-\frac{1}{18}+\frac{81}{2}+\frac{3}{2}\right)-\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{18}+\frac{1}{6}\right)\right]$$
$$=\frac{\pi}{2}\left(\frac{418}{9}-\frac{2}{9}\right)$$
$$=\frac{208\pi}{9}$$

7 $y^{\frac{2}{3}} = 4 - x^{\frac{2}{3}}$

$$\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{1}{3}}$$

$$y^{-\frac{1}{3}}\frac{dy}{dx} = -x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = -y^{\frac{1}{3}}x^{-\frac{1}{3}}$$

$$\left(\frac{dy}{dx}\right)^2 = y^{\frac{2}{3}}x^{-\frac{2}{3}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + y^{\frac{2}{3}}x^{-\frac{2}{3}}$$

$$= 1 + \left(4 - x^{\frac{2}{3}}\right)x^{-\frac{2}{3}}$$

$$= 1 + \left(4x^{-\frac{2}{3}} - 1\right)$$

$$= 4x^{-\frac{2}{3}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = 2x^{-\frac{1}{3}}$$

Using $S = 2\pi\int_{x_A}^{x_B} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ gives:

$$S = 2\pi\int_0^8 x \times 2x^{-\frac{1}{3}}\,dx$$

$$= 4\pi\int_0^8 x^{\frac{2}{3}}\,dx$$

$$= 4\pi\left[\frac{3}{5}x^{\frac{5}{3}}\right]_0^8$$

$$= \frac{12\pi}{5}\left(8^{\frac{5}{3}} - 0^{\frac{5}{3}}\right)$$

$$= \frac{384\pi}{5}$$

8 **a** $x^2 + y^2 = R^2$

$y^2 = R^2 - x^2$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{x}{y}$

$= -\frac{x}{\sqrt{R^2 - x^2}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{R^2 - x^2}$

$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{R^2 - x^2}}$

Using $S = 2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ gives:

$$S = 2\pi\int_{x_A}^{x_B} \sqrt{R^2 - x^2}\sqrt{1+\frac{x^2}{R^2 - x^2}}\,dx$$

$$= 2\pi\int_{-R}^{R} \sqrt{R^2 - x^2}\sqrt{\frac{R^2}{R^2 - x^2}}\,dx$$

$$= 2\pi\int_{-R}^{R} R\,dx$$

$$= 2\pi R[x]_{-R}^{R}$$

$$= 2\pi R(R-(-R))$$

$$= 4\pi R^2$$

b For the sphere:

$$2\pi\int_a^b R\,dx = 2\pi R[x]_a^b$$

$$= 2\pi R(b-a)$$

For the cylinder:
The length, $L = b - a$ and the circumference is $2\pi R$
Therefore, the surface area is $2\pi R(b-a)$
which is the same as for the sphere between a and b.

9 $y = 2at \Rightarrow \frac{dy}{dt} = 2a \Rightarrow \left(\frac{dy}{dt}\right)^2 = 4a^2$

$x = at^2 \Rightarrow \frac{dx}{dt} = 2at \Rightarrow \left(\frac{dx}{dt}\right)^2 = 4a^2t^2$

$$\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{4a^2t^2 + 4a^2}$$

$$= \sqrt{4a^2\left(t^2 + 1\right)}$$

$$= 2a\sqrt{t^2 + 1}$$

Using $S = 2\pi\int_{t_A}^{t_B} y\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$ gives:

$$S = 2\pi\int_0^2 2at \times 2a\sqrt{t^2 + 1}\,dt$$

$$= 8\pi a^2\int_0^2 t\sqrt{t^2 + 1}\,dt$$

$$= \frac{8\pi a^2}{3}\left[\left(t^2 + 1\right)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{8\pi a^2}{3}\left[\left(2^2 + 1\right)^{\frac{3}{2}} - \left(0^2 + 1\right)^{\frac{3}{2}}\right]$$

$$= \frac{8\pi a^2}{3}\left(5\sqrt{5} - 1\right) \text{ as required}$$

10 $y = \tanh t \Rightarrow \frac{dy}{dt} = \text{sech}^2 t \Rightarrow \left(\frac{dy}{dt}\right)^2 = \text{sech}^4 t$

$x = \text{sech}\, t \Rightarrow \frac{dx}{dt} = -\text{sech}\, t \tanh t \Rightarrow \left(\frac{dx}{dt}\right)^2 = \text{sech}^2 t \tanh^2 t$

$$\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{\text{sech}^4 t + \text{sech}^2 t \tanh^2 t}$$
$$= \sqrt{\text{sech}^2 t\left(\text{sech}^2 t + \tanh^2 t\right)}$$
$$= \text{sech}\, t$$

Using $S = 2\pi \int_{t_A}^{t_B} y \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$ gives:

$$S = 2\pi \int_0^{\ln 2} \tanh t\, \text{sech}\, t\, dt$$
$$= 2\pi\left[-\text{sech}\, t\right]_0^{\ln 2}$$
$$= -2\pi\left[\frac{1}{\frac{1}{2}\left(e^t + e^{-t}\right)}\right]_0^{\ln 2}$$
$$= -4\pi\left(\frac{1}{e^{\ln 2} + e^{-\ln 2}} - \frac{1}{e^0 + e^0}\right)$$
$$= -4\pi\left(\frac{1}{2 + \frac{1}{2}} - \frac{1}{1+1}\right)$$
$$= -4\pi\left(\frac{2}{5} - \frac{1}{2}\right)$$
$$= \frac{2\pi}{5}$$

11 a $y = 2t^3 \Rightarrow \frac{dy}{dt} = 6t^2 \Rightarrow \left(\frac{dy}{dt}\right)^2 = 36t^4$

$x = 3t^2 \Rightarrow \frac{dx}{dt} = 6t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 36t^2$

$$\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{36t^4 + 36t^2}$$

$$= \sqrt{36t^2\left(t^2+1\right)}$$

$$= 6t\sqrt{t^2+1}$$

Using $S = 2\pi\int_{t_A}^{t_B} x\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$ gives:

$$S = 2\pi\int_0^2 3t^2 \times 6t\sqrt{t^2+1}\,dt$$

$$= 36\pi\int_0^2 t^3\sqrt{t^2+1}\,dt \text{ as required}$$

11 b Let $u = t^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = 2t$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = t\left(t^2+1\right)^{\frac{1}{2}} \Rightarrow v = \frac{1}{3}\left(t^2+1\right)^{\frac{3}{2}}$$

$$S = 36\pi\left(\left[\frac{1}{3}t^2\left(t^2+1\right)^{\frac{3}{2}}\right]_0^2 - \frac{2}{3}\int_0^2 t\left(t^2+1\right)^{\frac{3}{2}}\,\mathrm{d}t\right)$$

$$= 36\pi\left(\left[\frac{1}{3}t^2\left(t^2+1\right)^{\frac{3}{2}}\right]_0^2 - \frac{2}{3}\left[\frac{1}{5}\left(t^2+1\right)^{\frac{5}{2}}\right]_0^2\right)$$

$$= 36\pi\left(\left[\frac{1}{3}(2)^2\left(2^2+1\right)^{\frac{3}{2}} - \frac{1}{3}(0)^2\left(0^2+1\right)^{\frac{3}{2}}\right]_0^2 - \frac{2}{15}\left[\left(2^2+1\right)^{\frac{5}{2}} - \left(0^2+1\right)^{\frac{5}{2}}\right]_0^2\right)$$

$$= 36\pi\left(\left[\frac{1}{3}(2)^2\left(2^2+1\right)^{\frac{3}{2}}\right] - \frac{2}{15}\left[\left(2^2+1\right)^{\frac{5}{2}} - 1\right]\right)$$

$$= 36\pi\left(\left[\frac{4}{3}\times 5^{\frac{3}{2}}\right] - \frac{2}{15}\left[5^{\frac{5}{2}} - 1\right]\right)$$

$$= 36\pi\left(\left[\frac{4}{3}\times 5^{\frac{3}{2}}\right] - \frac{2}{15}\left[5\times 5^{\frac{3}{2}} - 1\right]\right)$$

$$= 36\pi\left(\left[\frac{4}{3}\times 5^{\frac{3}{2}}\right] - \left[\frac{2}{3}\times 5^{\frac{3}{2}} - \frac{2}{15}\right]\right)$$

$$= 36\pi\left(\frac{2}{3}\times 5^{\frac{3}{2}} + \frac{2}{15}\right)$$

$$= \frac{24}{5}\pi\left(5\times 5^{\frac{3}{2}} + 1\right)$$

$$= \frac{24}{5}\pi\left(25\sqrt{5} + 1\right)$$

12 $y = t - \frac{1}{3}t^3 \Rightarrow \frac{dy}{dt} = 1 - t^2 \Rightarrow \left(\frac{dy}{dt}\right)^2 = 1 - 2t^2 + t^4$

$x = t^2 \Rightarrow \frac{dx}{dt} = 2t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 4t^2$

$$\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{1 - 2t^2 + t^4 + 4t^2}$$
$$= \sqrt{1 + 2t^2 + t^4}$$
$$= \sqrt{\left(1 + t^2\right)^2}$$
$$= 1 + t^2$$

Using $S = 2\pi\int_{t_A}^{t_B} y\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$ gives:

$$S = 2\pi\int_0^1 \left(t - \frac{1}{3}t^3\right)\left(1 + t^2\right)dt$$
$$= 2\pi\int_0^1 \left(t - \frac{1}{3}t^3 + t^3 - \frac{1}{3}t^5\right)dt$$
$$= 2\pi\left[\frac{1}{2}t^2 - \frac{1}{12}t^4 + \frac{1}{4}t^4 - \frac{1}{18}t^6\right]_0^1$$
$$= 2\pi\left(\frac{1}{2} - \frac{1}{12} + \frac{1}{4} - \frac{1}{18}\right)$$
$$= \frac{11\pi}{9}$$

13 $y = a\sin^3 t \Rightarrow \frac{dy}{dt} = 3a\sin^2 t\cos t \Rightarrow \left(\frac{dy}{dt}\right)^2 = 9a^2\sin^4 t\cos^2 t$

$x = a\cos^3 t \Rightarrow \frac{dx}{dt} = -3a\cos^2 t\sin t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9a^2\cos^4 t\sin^2 t$

$$\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = \sqrt{9a^2\sin^4 t\cos^2 t + 9a^2\cos^4 t\sin^2 t}$$

$$= \sqrt{9a^2\sin^2 t\cos^2 t\left(\sin^2 t + \cos^2\right)}$$

$$= 3a\sin t\cos t$$

Using $S = 2\pi\int_{t_A}^{t_B} y\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$ gives:

$$S = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} a\sin^3 t \times 3a\sin t\cos t\,dt$$

$$= 6a^2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^4 t\cos t\,dt$$

$$= 6a^2\pi\left[\frac{1}{5}\sin^5 t\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{6a^2\pi}{5}\left[\sin^5\left(\frac{\pi}{2}\right) - \sin^5\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{6a^2\pi}{5}\left[1^5 - \left(\frac{1}{2}\right)^5\right]$$

$$= \frac{6a^2\pi}{5}\left(\frac{31}{32}\right)$$

$$= \frac{93a^2\pi}{80}$$

14 $y = e^x \Rightarrow \frac{dy}{dx} = e^x \Rightarrow \left(\frac{dy}{dx}\right)^2 = e^{2x}$

Using $S = 2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ gives:

$$S = 2\pi\int_0^{\ln 2} e^x\sqrt{1+e^{2x}}\,dx$$

Let $u = e^x \Rightarrow du = e^x dx$
When $x = 0$, $u = 1$
When $x = \ln 2$, $u = 2$

$$S = 2\pi\int_1^2 \sqrt{1+u^2}\,du$$

Let $u = \sinh\theta \Rightarrow du = \cosh\theta\,d\theta$
When $u = 1$, $\theta = \text{arsinh}(1)$
When $u = 2$, $\theta = \text{arsinh}(2)$

$$S = 2\pi\int_{\text{arsinh}(1)}^{\text{arsinh}(2)} \sqrt{1+\sinh^2\theta}\times\cosh\theta\,d\theta$$
$$= 2\pi\int_{\text{arsinh}(1)}^{\text{arsinh}(2)} \sqrt{\cosh^2\theta}\times\cosh\theta\,d\theta$$
$$= 2\pi\int_{\text{arsinh}(1)}^{\text{arsinh}(2)} \cosh^2\theta\,d\theta$$
$$= 2\pi\int_{\text{arsinh}(1)}^{\text{arsinh}(2)} \frac{\cosh 2\theta+1}{2}\,d\theta$$
$$= \pi\int_{\text{arsinh}(1)}^{\text{arsinh}(2)} (\cosh 2\theta+1)\,d\theta$$
$$= \pi\left[\frac{1}{2}\sinh 2\theta+\theta\right]_{\text{arsinh}(1)}^{\text{arsinh}(2)}$$
$$= \pi\left[\sinh\theta\cosh\theta+\theta\right]_{\text{arsinh}(1)}^{\text{arsinh}(2)}$$
$$= \pi\left[\sinh\theta\sqrt{1+\sinh^2\theta}+\theta\right]_{\text{arsinh}(1)}^{\text{arsinh}(2)}$$
$$= \pi\left[2\sqrt{1+2^2}+\text{arsinh}(2)-1\sqrt{1+1^2}-\text{arsinh}(1)\right]$$
$$= \pi\left(\text{arsinh}(2)-\text{arsinh}(1)+2\sqrt{5}-\sqrt{2}\right) \text{ as required}$$